

Quantum anticyclifugal force for wormhole geometry

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Abstract

We show the existence of an anticyclifugal force in a wormhole geometry in R^3 . This counterintuitive force was shown to exist in a flat R^2 space. The role the geometry plays in the appearance of this force is discussed.

Key words: quantum anticyclifugal force, wormhole geometry

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Dimensionality of space plays a very important role in quantum mechanics. Quantum mechanics in flat Euclidean space in one, two and three dimensions leads to very different behavior of a quantum particle.

Especially quantum mechanics in a flat two dimensional Euclidean space R^2 gives very unexpected results like e.g. the quantum anticyclifugal force for waves with zero angular momentum [1] [2] [3]. This quantum anticyclifugal force is part of the so called quantum fictitious forces that appear in two and three space dimensions [4]. This phenomenon is due on one hand to the

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commutator of the radial momentum p_r and the unit vector in radial direction $\frac{\vec{r}}{r}$ and on the other hand to the renormalization of the wave function so that the wave function is normalized in flat space.

Constraining a particle to move on a two-dimensional surface requires in general a special treatment for the Schrödinger equation. It is not physical to suppose that the surface has no thickness at all (e.g. because of the Heisenberg uncertainty principle). If on the contrary the particle is allowed to move on a surface with finite thickness and then let the thickness go to zero, then there will appear an effective potential in the Schrödinger equation for a particle on a curved surface: this potential depends on the mean \mathbf{M} and the Gaussian curvature \mathbf{K} of the surface [5]. If the surface is flat then there is no additional potential.

Quantum mechanics in three dimensions of a curved space so far has not attracted much attention. It has been shown that a spherical wave (s -wave) always blows up in three dimensions (with no external potential). In two dimensions, as we have already mentioned, on the contrary there is a very counterintuitive collapsing of a s -wave to the origin. As the potential is attractive, in order to get localized states around the origin of the coordinate system one has to add a Dirac- δ function potential there[1]. Another more natural way to get this is to deform the surface (creating e.g. a gaussian bump centered at the origin of the coordinate system) In this case the curvature effect of the bump superposes with the anticentrifugal potential for s -waves and create a natural setup for localized states around the origin of the coordinate system [6].

In this paper we study the effect of curvature of a 3 dimensional space on the

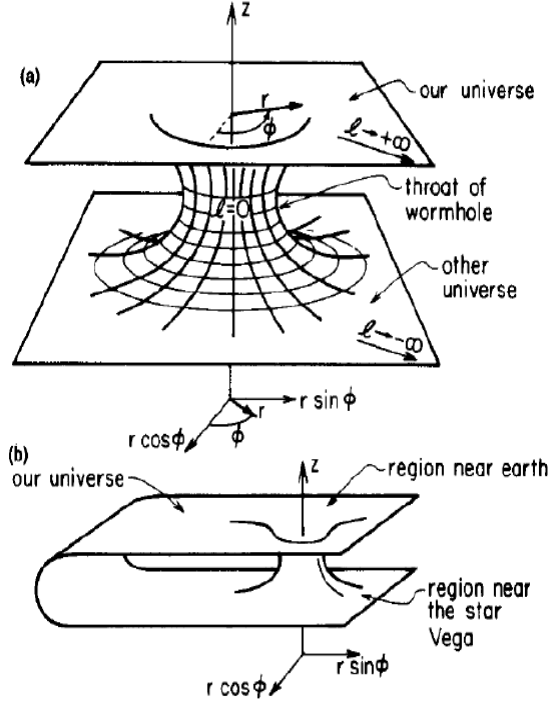


Fig. 1. Schematic representation of a wormhole geometry adapted from [7]

ant centrifugal force and show that for a suitable geometry [7] the quantum ant centrifugal potential is present in three dimensions too. This is true for s -waves and for higher angular momenta there are metastable states. The geometry is given by the following element of length[7]:

$$ds^2 = -c^2 dt^2 + dl^2 + (b^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

where the coordinates belong to the following intervals: $t \in [-\infty, +\infty]$, $l \in [-\infty, +\infty]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ and b is the shape function of the wormhole (in general $b = b(l)$ and for $l = 0$, $b = b(0) = b_0 = \text{const}$ represents the radius of the throat of the wormhole). c is the speed of light. In this metric t measures proper time of a static observer; l is a radial coordinate measuring proper radial distance at constant time; θ and ϕ are spherical polar coordinates. We are interested in the case $t = \text{const}$ in Eq.(1). This leads to

the following metric we will use from now on:

$$ds^2 = dl^2 + (b^2(l) + l^2)(d\theta^2 + \sin^2 \theta d\phi^2) \quad (2)$$

$$dl^2 + f^2(l)d\Omega_2 \quad (3)$$

where $f^2(l) = (b^2(l) + l^2)$ and $d\Omega_2 = d\theta^2 + \sin^2 \theta d\phi^2$. Let us note here that for $b = 0$ we get the usual metric for a flat R^3 space where l plays the role of a radial coordinate. The space we are considering is shown schematically on fig.(1).

In general curvilinear coordinates the stationary Schrödinger equation $-\frac{\hbar^2}{2m}\Delta\psi = E\psi$ is given by the following expression:

$$-\frac{\hbar^2}{2m} \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial q_2} \right) \right. \quad (4)$$

$$\left. + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right] = E\psi \quad (5)$$

where h_1 , h_2 and h_3 are the Lamé coefficients. In our case $h_1 = h_l = 1$, $h_2 = h_\theta = \sqrt{b^2(l) + l^2}$ and $h_3 = h_\phi = \sqrt{b^2(l) + l^2} \sin \theta$. The metric determinant $g = (b^2(l) + l^2)^2 \sin^2 \theta$. Then the Laplacian becomes:

$$\Delta\psi = \frac{1}{(b^2 + l^2) \sin \theta} \left[(b^2 + l^2) \sin \theta \frac{\partial^2 \psi}{\partial l^2} + \frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right. \quad (6)$$

$$\left. 2 \sin \theta (bb' + l) \frac{\partial \psi}{\partial l} + \sin \theta \frac{\partial^2 \psi}{\partial \theta^2} + \cos \theta \frac{\partial \psi}{\partial \theta} \right] \quad (7)$$

In the above expression b stands for $b(l)$ and b' stands for $\frac{db}{dl}$. From now on we will note b instead of $b(l)$. Now because we want to normalize the radial wave function with a flat norm we introduce the following ansatz as it is usually

done in R^3 :

$$\psi = \frac{\Phi}{\sqrt{(b^2 + l^2)}} \quad (8)$$

After some algebra we get the following Schrödinger equation for the wave function Φ :

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial l^2} + V_{eff} \Phi + \frac{1}{(b^2 + l^2)} \hat{M}^2 \Phi = E \Phi \quad (9)$$

Where

$$V_{eff} = \frac{b^3 b'' + b^2 + b'^2 l^2 + b b'' l^2 - 2 b b' l}{(b^2 + l^2)^2} \quad (10)$$

In eq.(6) \hat{M}^2 represents the usual angular momentum operator. Now we separate variables in "radial" and angular and set $\Phi(l, \theta, \phi) = \Phi_1(l) \Phi_2(\theta, \phi)$. Note that as usual $\hat{M}^2 \Phi_2 = L(L+1) \Phi_2$. The "radial" Schrödinger equation now reads:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Phi_1}{\partial l^2} + \frac{L(L+1)}{b^2 + l^2} + V_{eff} \Phi_1 = E \Phi_1 \quad (11)$$

Let us first consider the case $b = b_0 = \text{const}$. In this case the effective potential reduces to $V_{eff} = \frac{b_0^2}{(b_0^2 + l^2)^2}$. This potential is repulsive and for $b_0 = 0$ we recognize the usual "radial" Schrödinger equation in R^3 . For waves with angular momentum zero i.e. $L = 0$ we get an overall repulsive effective potential:

$$V_{eff} = \frac{\hbar^2}{2m} \frac{b_0^2}{(b_0^2 + l^2)^2} \quad (12)$$

contrary to what happens in two dimensional flat space R^2 . For all other $L \neq 0$ the overall effective potential V_{eff} is also repulsive.

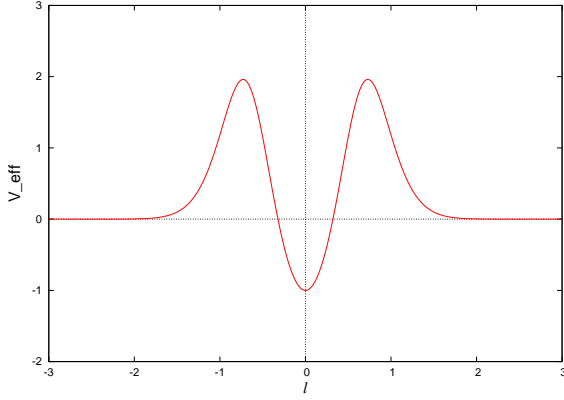


Fig. 2. V_{eff} for $b = b_0 e^{-l^2/b_0^2}$ and $L=0$ (we have set $b_0 = 1$ in arbitrary units)

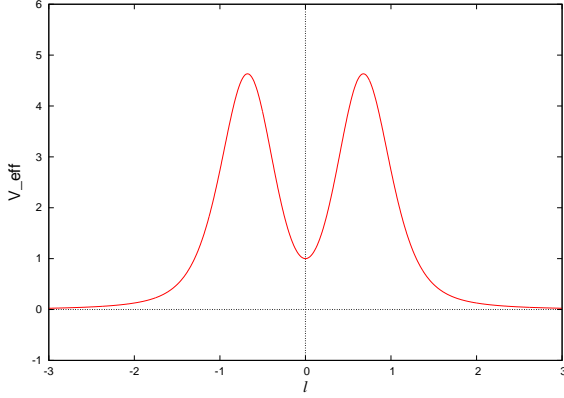


Fig. 3. V_{eff} for $b = b_0 e^{-l^2/b_0^2}$ and $L=1$ (we have set $b_0 = 1$ in arbitrary units)

Now let us consider a more general case of a shape function $b = b_0 e^{-l^2/b_0^2}$ where $b_0 = const$. For very large l the space is almost Euclidean and for small l the radius of the throat is almost b_0 . Now the effective potential V_{eff} has the following nontrivial form:

$$V_{eff} = \frac{1}{(b_0^2 e^{-2l^2/b_0^2} + l^2)^2} \left[-2b_0^2 e^{-4l^2/b_0^2} + b_0^2 e^{-2l^2/b_0^2} \right. \quad (13)$$

$$\left. + 2l^2 e^{-2l^2/b_0^2} + 4l^2 e^{-4l^2/b_0^2} + \frac{8l^4}{b_0^2} e^{-2l^2/b_0^2} \right] \quad (14)$$

This effective potential is represented schematically in Fig.2 for $b_0 = 1$ in

arbitrary units. The depth of the potential hole at the origin (at the center of the throat of the wormhole where $l = 0$) is $V_{eff}(0) = -\frac{1}{b_0^2}$. There are obviously localized states at the origin. Contrary to what happens in R^2 there is no need to add a potential at the origin in order to create bound states - here it is the geometry that creates them. In general in this case the potential is more complicated than the R^2 case where the potential is purely attractive.

Now, let us consider the case for $L \neq 0$. Contrary to what happens in R^2 or in R^3 for $b = b_0$ where the potential for $L \neq 0$ is purely repulsive, in this case there are metastable states at the origin, close to the throat. In Fig.2 V_{eff} is shown for angular momentum $L = 1$.

It is possible to give a partial qualitative explanation of this phenomenon. As the quantum effective potential depends on the underlying geometry it should be sensitive to any stretching of the manifold. According to the Heisenberg uncertainty relation $\Delta q \Delta p_q = \hbar$ where q is some generalized coordinate of the manifold and p_q is the corresponding momentum. If in some place the manifold is stretched Δq is bigger than the corresponding distance between the same points in the flat embedding manifold and therefore Δp_q is smaller and hence the energy ($E = \frac{\hbar^2(\Delta p_q)^2}{2m}$) is lower.

On Fig.3 we see that the "radius" of $d\Omega_2$ has a local maximum at the origin and obviously the manifold is "stretched" there which corresponds to a minimim of the effective potential. Similar behavior is seen in the one and two dimensional cases [6,8,9].

We have shown that the attractive quantum effective potential can appear in three dimensions too in case of a "wormhole-geometry" when the shape function $b(l)$ has a local maximum for $l = 0$. For higher angular momenta

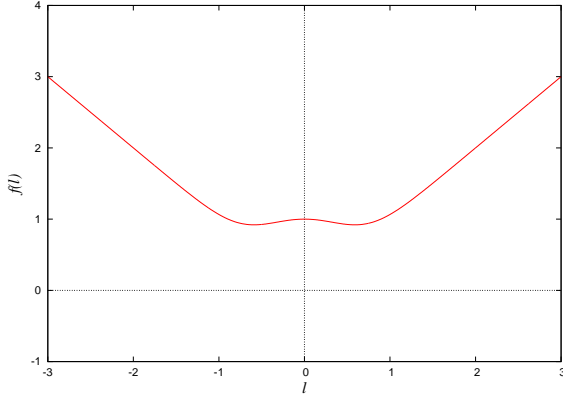


Fig. 4. the "radius" $f(l)$ of $d\Omega_2$ in eq(2) (we have set $b_0 = 1$ in arbitrary units) the corresponding quantum effective potential is also nontrivial and shows metastable states at the throat of the wormhole.

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